J. Appl. Maths Mechs Vol. 57, No. 1, pp. 159–170, 1993
Printed in Great Britain.

0021-8928/93 $24.00+.00

A CONTACT PROBLEM WITH FRICTION AND ADHESION FOR AN ELASTIC LAYER WITH STIFFENERS†

YU. A. ANTIPOV and N. KH. ARUTYUNYAN⊕

Odessa, Moscow

(*Received* 14 *July* 1992)

The contact of an elastic layer with an infinite stiffener to which a uniform constant normal load and a concentrated tangential force are applied, is considered. In the neighbourhood of the point of application of this force, on the line of the contact of the stiffener with the layer a segment is separated out, on which the effect of the Coulomb friction is taken into account. Outside this segment the stiffener and the layer are under conditions of complete adhesion. The problem is reduced to a Prandtl-type integro-differential equation specified on two semi-infinite segments, for whose solution an analytical method is proposed. The method is based on reducing the equation to a vectorial Riemann problem and then to an algebraic Poincaré–Koch system. The latter admits of an explicit solution and also inversion through recurrent relations that are effective when using numerical computations. The length of the Coulomb friction zone and the contact tangential stresses in the adhesion zone are determined. Unlike Melan's problem [1] the contact stresses have no logarithmic singularity and are continuous everywhere in the contact area.

The solution of the problem of the contact of a layer with a finite stiffener subject to a uniform pressure along the whole length and to an extension by forces concentrated at the tips is also obtained. The contact area is divided into an intermediate zone of adhesion and two zones of coulomb friction. The problem is reduced to a Prandtl-type integro-differential equation specified on the segment, and it is solved by analogy with the solution of the equation of the first problem. Such a formulation of the problem implies that the contact tangential stresses are bounded at the tips of the stiffener and are continuous at the points of the boundary between the zones of adhesion and Coulomb friction. When adhesion occurs along the entire line of contact the tangential stresses, in general, have a root singularity [2]. In the problem of the contact of a plane punch with a half-plane under conditions of friction and adhesion, the contact stresses at the tips of the punch have [3] a power singularity (that differs from a root one).

1. THE CONTACT PROBLEM FOR AN ELASTIC LAYER WITH AN INFINITE STIFFENER UNDER CONDITIONS OF FRICTION AND ADHESION

LET AN infinite stringer $\{-\infty<x<+\infty,\ 0<y<h\}$ with modulus of elasticity E_0 and Poisson's ratio ν_0 be attached to an elastic layer $\{-\infty<x<+\infty,\ -b<y<0\}$ whose elastic parameters are E and D, respectively (Fig. 1). The plane deformation of the elastic layer linked to the rigid foundation is examined. The normal uniform load of the intensity p and the concentrated tangential force T (its point of application is $x=0$) act upon the stiffener. The contact area is divided into the adhesion zones $x<-a$ and $x>a$ and the Coulomb friction zone $|x|<a$. It is required to find the location of the point a and the distribution of the contact tangential stresses $\tau(x)$ over the border line between the stringer and the layer.

†*Prikl. Mat. Mekh.* Vol. 57, No. 1, pp. 137–147, 1993.

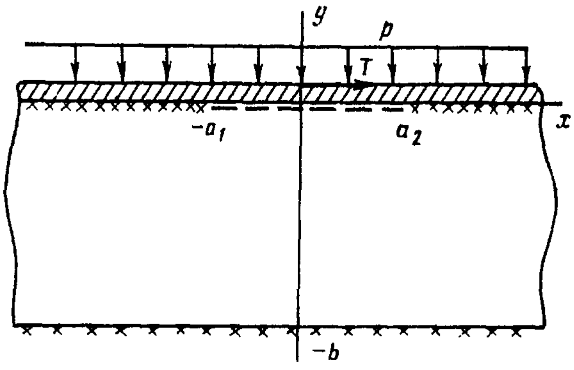

Fig. 1.

The equation of equilibrium of the stringer leads to the following expression for its axial deformation

$$\epsilon_x^0(x) = \frac{1-\nu_0^2}{hE_0}\left[\int_{-\infty}^{x} \tau(\xi)\,d\xi - TH(x)\right] \tag{1.1}$$

where $H(x)$ is the Heaviside function. The equilibrium of the stringer ensures the condition

$$\int_{-\infty}^{\infty} \tau(x)\,dx = T \tag{1.2}$$

Consider the problem of plane deformation of the elastic strip

$$\begin{aligned} &\Delta^2 U(x,y) = 0, \ |x| < \infty, \ -b < y < 0 \\ &\sigma_y|_{y=0} = \sigma(x), \ \tau_{xy}|_{y=0} = \tau(x), \ |x| < \infty \\ &u|_{y=-b} = 0, \ v|_{y=-b} = 0, \ |x| < \infty \end{aligned} \tag{1.3}$$

Here U is the Airy function, $\|u, v\|$ is the vector of displacements, and σ_y and τ_{xy} are the components of the stress tensor. By using a Fourier transformation we obtain the following relation for the horizontal deformations of the strip for $y=0$

$$\begin{aligned} \epsilon_x(x,0) = \frac{1+\nu}{2\pi E} \int_{-\infty}^{\infty} &[(\kappa\kappa_- \mathrm{sh}^2\alpha b - \alpha^2 b^2)\,\sigma_\alpha^0 - \kappa_+(\kappa\,\mathrm{sh}\alpha b\,\mathrm{ch}\alpha b + \alpha b)\,i\tau_\alpha^0] \times \\ &\times [d(\alpha)]^{-1} e^{-i\alpha x}\,d\alpha, \quad d(\alpha) = \kappa\,\mathrm{sh}^2\alpha b + \alpha^2 b^2 + \kappa_+^2 \end{aligned} \tag{1.4}$$

$$\kappa_\pm = \tfrac{1}{2}(\kappa \pm 1), \quad \kappa = 3 - 4\nu, \quad \|\sigma_\alpha^0, \tau_\alpha^0\| = \int_{-\infty}^{\infty} \|\sigma(x), \tau(x)\|\, e^{i\alpha x}\,dx$$

We have $\sigma(x) = -p$, $|x| < \infty$ since the stiffener has no bending rigidity. The tangential and normal stresses are related by the condition (μ is the coefficient of friction)

$$\tau(x) = \mu p, \quad |x| < a$$

in the slippage zone, but we have $\tau(x) < \mu p$, $|x| > a$ in the adhesion zones since the tangential contact stresses are insufficient to cause slippage. Moreover, the horizontal deformations $\epsilon_x^0(x)$ and $\epsilon_x(x,0)$ of the stiffener and of the strip have to be identical, i.e.

$$\epsilon_x^0(x) = \epsilon_x(x,0), \quad |x| > a \tag{1.5}$$

The stresses at the point $x=a$ have to be bounded.

We substitute expressions (1.1) and (1.4) into relation (1.5), take into account that $\sigma_\alpha^0 = -2\pi p\delta(\alpha)$ and introduce the new unknown function

$$\chi(t) = bT^{-1}\tau(-a+bt) \tag{1.6}$$

As the result, we obtain the following Prandtl-type integro-differential equation

$$\int_{-\infty}^{t} \chi(\tau)d\tau + \gamma \int_{-\infty}^{\infty} S(t-\tau)\chi(\tau)d\tau = H\left(t-\frac{\lambda}{2}\right) \tag{1.7}$$

$$t \in (-\infty, 0)\cup(\lambda, \infty),\ \gamma = 2hE_0(1-\nu^2)b^{-1}E^{-1}(1-\nu_0^2)^{-1},\ \lambda = 2ab^{-1} \tag{1.8}$$

$$S(t) = \frac{1}{\pi}\int_0^{\infty} \frac{\kappa \operatorname{sh}\alpha \operatorname{ch}\alpha + \alpha}{\kappa \operatorname{sh}^2\alpha + \alpha^2 + \kappa_+^2} \sin\alpha t\, d\alpha,\quad \chi(t) = p_0\ (0<t<\lambda),\ p_0 = \frac{\mu p b}{T}$$

We will seek a solution of Eq. (1.7) in the set of functions bounded at the point $t=0$ and $t=\lambda$. Let us introduce the one-sided functions

$$\chi_-(t) = \begin{cases} \chi(t), & t<0 \\ 0, & t>0 \end{cases},\quad \chi_+(t) = \begin{cases} \chi(t), & t>\lambda \\ 0, & t<\lambda \end{cases}$$

$$f_0(t) = \begin{cases} 1, & 0<t<\lambda \\ 0, & t \notin (0,\lambda) \end{cases},\quad \chi(t) = \chi_-(t) + \chi_+(t) + p_0 f(t),\ |t|<\infty \tag{1.9}$$

and extend the definition of Eq. (1.7) over the entire real axis using the function $\chi_0(t)$ which is unknown in the interval $0<t<\lambda$ and vanishes outside this interval. We have

$$\int_{-\infty}^{t} \chi(\tau)d\tau + \gamma \int_{-\infty}^{\infty} S(t-\tau)\chi(\tau)d\tau = H\left(t-\frac{\lambda}{2}\right) + \chi_0(t),\ |t|<\infty$$

Consider the Fourier transforms

$$\Phi_1^+(\alpha) = \int_0^{\lambda} \chi_0(t)e^{i\alpha t}dt,\quad \Phi_1^-(\alpha) = \int_{-\lambda}^{0} \chi_0(\lambda+t)e^{i\alpha t}dt$$

$$\Phi_2^+(\alpha) = \int_0^{\infty} \chi(\lambda+t)e^{i\alpha t}dt,\quad \Phi_2^-(\alpha) = \int_{-\infty}^{0} \chi(t)e^{i\alpha t}dt \tag{1.10}$$

The functions $\Phi_1^\pm(\alpha)$ are integer and $\Phi_2^\pm(\alpha)$ are analytic in $C^\pm$: $\operatorname{Im}\alpha \gtrless 0$. Furthermore, if we take into account [4] the formulae

$$V[H(t)] = \int_{-\infty}^{\infty} H(t)e^{i\alpha t}dt = -\frac{i}{\alpha+i0}$$

$$V\left[\int_{-\infty}^{t} \chi(\tau)d\tau\right] = -\frac{i}{\alpha+i0}\left[\Phi_2^-(\alpha) + e^{i\alpha\lambda}\Phi_2^+(\alpha) + \frac{p_0}{i\alpha}\left(e^{i\alpha\lambda}-1\right)\right]$$

and, by virtue of (1.10), the relation

$$\Phi_1^+(\alpha) = e^{i\alpha\lambda}\Phi_1^-(\alpha)$$

we obtain the vectorial Riemann problem

$$G(\alpha)\left[\Phi_2^\pm(\alpha) + e^{\mp i\alpha\lambda}\Phi_2^\mp(\alpha) \pm \frac{p_0}{i\alpha}\left(1-e^{\mp i\alpha\lambda}\right)\right] = -i\alpha\Phi_1^\mp(\alpha) + e^{\mp i\alpha\lambda/2}$$

$$-\infty < \alpha < +\infty \tag{1.11}$$
$$G(\alpha) = 1 + \gamma\alpha(\kappa \operatorname{sh}\alpha \operatorname{ch}\alpha + \alpha)(\kappa \operatorname{sh}^2\alpha + \alpha^2 + \kappa_+^2)^{-1}$$

The factorization of the function $G(\alpha)$ is defined by the relations

$$G(\alpha) = K^+(\alpha)X^+(\alpha)K^-(\alpha)X^-(\alpha),\ X^\pm(\alpha) = [X(\alpha)]^{\pm 1},\ \alpha \in C^\pm \tag{1.12}$$
$$K^\pm(\alpha) = (\pi\gamma)^{1/2} \frac{\Gamma(1 \mp i\alpha/\pi)}{\Gamma(1/2 \mp i\alpha/\pi)},\quad X(\alpha) = \exp\left(\frac{\alpha}{\pi i} \int_0^\infty \ln G_0(\eta) \cdot \frac{d\eta}{\eta^2 - \alpha^2}\right) (\alpha \neq 0)$$
$$G_0(\alpha) = 1 + \frac{\operatorname{th}\alpha}{\gamma\alpha} + \frac{\alpha \operatorname{th}\alpha - \alpha^2 - \kappa_+^2}{\kappa \operatorname{sh}^2\alpha + \kappa_+^2 + \alpha^2},\quad X^\pm(0) = \gamma^{-1/2}$$

We substitute the first formula of (1.12) into the boundary condition (1.11) and obtain

$$K^\pm(\alpha)X^\pm(\alpha)[\Phi_2^\pm(\alpha) \pm (i\alpha)^{-1}p_0] \mp (i\alpha)^{-1}p_0 = [K^\mp(\alpha)X^\mp(\alpha)]^{-1}\{-i\alpha\Phi_1^\mp(\alpha) +$$
$$+ e^{\mp i\alpha\lambda/2} - e^{\mp i\alpha\lambda}G(\alpha)[\Phi_2^\mp(\alpha) \mp (i\alpha)^{-1}p_0]\} \mp (i\alpha)^{-1}p_0 \tag{1.13}$$

In $C^\pm$ the left-hand side of the last equality is analytic and the right-hand side has a denumerable set of poles that are identical with the poles $\alpha = \mp i\beta_m$ $(m = 1, 2, \ldots)$ of the function $G(\alpha)$. Here β_1 is the real root of the function

$$h(\beta) = \kappa_+^2 - \beta^2 - \kappa \sin^2\beta$$

and $\beta_{2m+j} = 1/2[b_m - (-1)^j a_m i]$ $(m = 1, 2, \ldots;\ j = 0, 1)$, and the quantities $z_m = a_m + ib_m$ are the roots of the equation [5]

$$2\kappa \operatorname{ch} z + z^2 + \kappa^2 + 1 = 0$$

that are evaluated according to the iterative formula

$$z_n^{(k)} = 2\pi n i + \ln\varphi(z_n^{(k-1)})\ (k = 2, 3, \ldots),\quad z_n^{(1)} = 2\pi n i \tag{1.14}$$
$$\varphi(z) = -\kappa^{-1}(z^2 + \kappa^2 + 1) - e^{-z}$$

Making use of the functions

$$\Psi^\pm(\alpha) = \sum_{n=1}^\infty \frac{iA_n^\pm}{\alpha \pm i\beta_n}$$

we neutralize the poles on the right-hand sides of equalities (1.13) in $C^\pm$. Taking account of the fact that the tangential contact stresses $\tau(x)$ are bounded at the points $x = \pm a$ and making use of (1.13) we find the solution of problem (1.11)

$$\Phi_1^+(\alpha) = \frac{i}{\alpha}\{-e^{i\alpha\lambda/2} + K^+(\alpha)X^+(\alpha)[\Psi^-(\alpha) - \frac{p_0}{i\alpha}] + \frac{e^{i\alpha\lambda}G(\alpha)}{K^-(\alpha)X^-(\alpha)} \times$$
$$\times [\Psi^+(\alpha) + (i\alpha)^{-1}p_0]\},\quad \Phi_1^-(\alpha) = e^{-i\alpha\lambda}\Phi_1^+(\alpha) \tag{1.15}$$
$$\Phi_2^+(\alpha) = -\frac{p_0}{i\alpha} + \frac{\Psi^+(\alpha) + (i\alpha)^{-1}p_0}{K^+(\alpha)X^+(\alpha)},\quad \Phi_2^-(\alpha) = \frac{p_0}{i\alpha} + \frac{\Psi^-(\alpha) - (i\alpha)^{-1}p_0}{K^-(\alpha)X^-(\alpha)}$$

For the functions $\Phi_1^+(\alpha)$ and $\Phi_1^-(\alpha)$ to be analytic at the point $\alpha=0$ it is necessary and sufficient that the condition

$$\{-e^{i\alpha\lambda/2}+K^+(\alpha)X^+(\alpha)[\Psi^-(\alpha)-(i\alpha)^{-1}p_0]+e^{i\alpha\lambda}K^-(\alpha)X^-(\alpha)\times$$
$$\times[\Psi^+(\alpha)+(i\alpha)^{-1}p_0]\}_{\alpha=0}=0 \tag{1.16}$$

should be satisfied.

If we take into account the expansions in the neighbourhood of the point $\alpha=0$

$$G(\alpha)=1+O(\alpha^2),\ K^+(\alpha)=\gamma^{1/2}[1-i\alpha\pi^{-1}2\ln 2+O(\alpha^2)]$$
$$X(\alpha)=\gamma^{-1/2}[1-i\alpha d_0+O(\alpha^2)],\ \alpha\to 0$$
$$d_0=\frac{1}{\pi}\int_0^\infty \frac{G_0'(\eta)}{G_0(\eta)}\frac{d\eta}{\eta} \tag{1.17}$$

we obtain

$$\Psi^+(0)+\Psi^-(0)+(\lambda+2d_0+4\pi^{-1}\ln 2)p_0=1 \tag{1.18}$$

from (1.16)

Now let us consider the condition of equilibrium (1.2) of the stringer. Making use of (1.6), (1.9) and (1.10) we have

$$\Phi_2^-(0)+\Phi_2^+(0)+p_0\lambda=1 \tag{1.19}$$

instead of (1.2)

On account of (1.17), the substitution of formulae (1.15) into equality (1.19) leads to relation (1.18). Thus, condition (1.16) for the functions $\Phi_1^\pm(\alpha)$ to be analytic and condition (1.19) of equilibrium are identical and lead to (1.18).

The functions $\Phi_1^+(\alpha)$ and $\Phi_1^-(\alpha)$ given in (1.15) have poles at the points $i\beta_n$ and $-i\beta_n$ $(n=1, 2,\ldots)$ of the half-planes C^+ and C^-, respectively. In order to eliminate the poles it is necessary and sufficient that the conditions

$$\operatorname*{res}_{\alpha=\pm i\beta_n}\{e^{\pm i\alpha\lambda}G(\alpha)[K^\pm(\alpha)X^\pm(\alpha)]^{-2}[\Psi^\pm(\alpha)\pm(i\alpha)^{-1}p_0]+\Psi^\mp(\alpha)\}=0$$
$$(n=1,2,\ldots)$$

should be satisfied. These conditions are equivalent to the following infinite Poincaré–Koch system of algebraic equations

$$A_n^\pm=-\Delta_n e^{-\lambda\beta_n}\left(\pm\frac{p_0}{\beta_n}+\sum_{m=1}^\infty\frac{A_m^\mp}{\beta_n+\beta_m}\right)\ (n=1,2,\ldots)$$

$$\Delta_n=\frac{\gamma\beta_n}{2K_n^2X_n^2},\ K_n=(\pi\gamma)^{1/2}\frac{\Gamma(1+\pi^{-1}\beta_n)}{\Gamma(1/2+\pi^{-1}\beta_n)}\qquad X_n=\exp\left\{\frac{\beta_n}{\pi}\int_0^\infty \ln G_0(x)\frac{dx}{x^2+\beta_n^2}\right\} \tag{1.20}$$

which obviously yields the relation $A_n^+=A_n^-=A_n$. The coefficients A_n satisfy the system

$$A_n=\Delta_n e^{-\lambda\beta_n}\left(-\frac{p_0}{\beta_n}+\sum_{m=1}^\infty\frac{A_m}{\beta_n+\beta_m}\right)\ (n=1,2,\ldots)$$

that admits of the solution specified by the rapidly converging recurrent relations

$$A_n = e^{-\lambda\beta_n} \sum_{k=1}^{\infty} a_{nk}, \quad a_{n0} = -\frac{\Delta_n p_0}{\beta_n} \qquad a_{np} = \Delta_n \sum_{j=1}^{p} \frac{a_{j,p-j}}{\beta_n + \beta_j} e^{-\lambda\beta_j}$$

and also admits of the explicit solution

$$a_{np} = -\Delta_n p_0 \left[\frac{\Delta_p}{(\beta_n + \beta_p)\beta_p} + \frac{2}{\beta_n + \beta_1}\left(\frac{\Delta_1 e^{-\lambda\beta_1}}{2\beta_1}\right)^p + \right.$$

$$\left. + \sum_{m=1}^{p-2} \sum_{j_1=1}^{\sigma(0)-1} h_1 \sum_{j_2=1}^{\sigma(1)-1} h_2 \ldots \sum_{j_m=1}^{\sigma(m-1)-1} \frac{h_m}{\beta_{\sigma(m)}} \frac{\Delta_{\sigma(m)}}{\beta_{j_m} + \beta_{\sigma(m)}}\right]$$

$$h_m = \frac{\Delta_{j_m}}{\beta_{j_{m-1}} + \beta_{j_m}} \exp(-\lambda\beta_{j_m}), \quad \sigma_m = p - j_1 - \ldots - j_m, \quad \sigma(0) = p, \; j_0 = n$$

The parameter λ (and, of course, $a = \frac{1}{2}\lambda b$) is taken from condition (1.18) which is transformed into the equality

$$2 \sum_{n=1}^{\infty} \frac{A_n}{\beta_n} + (\lambda + 2d_0 + 4\pi^{-1}\ln 2)p_0 = 1$$

Since $A_n = A_n(\lambda)$ the last equality is a transcendental equation in λ.

We will obtain an expression for the tangential contact stresses when $|x| > a$ (if $|x| < a$ then $\tau(x) = \mu p$). By virtue of (1.6) and (1.10), we have

$$\tau(x) = Tb^{-1}\chi(b^{-1}(x + a))$$

$$\chi(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \left\{ \frac{p_0}{i\alpha} + \frac{K^+(\alpha)X^+(\alpha)}{G(\alpha)} \left[\Psi^-(\alpha) - \frac{p_0}{i\alpha}\right] \right\} e^{-i\alpha t} d\alpha, \quad t < 0 \qquad (1.21)$$

$$\chi(\lambda + t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \left\{ -\frac{p_0}{i\alpha} + \frac{K^-(\alpha)X^-(\alpha)}{G(\alpha)} \left[\Psi^+(\alpha) + \frac{p_0}{i\alpha}\right] \right\} e^{-i\alpha t} d\alpha, \quad t > 0$$

Denote the zeros of the function $G(\alpha)$ in the domain $C^\pm$ by $\pm i\delta_n$ $(n = 1, 2, \ldots)$. The first root δ_1 is real, but the other roots are complex

$$\delta_{2m+j} = \tfrac{1}{2}[d_m - ic_m(-1)^j] \quad (m = 1, 2, \ldots; \; j = 0, 1)$$

The numbers $z_m = c_m + id_m$ are evaluated by iterative formula (1.14) in which the function $\varphi(z)$ should be taken as

$$\varphi(z) = (1 + \tfrac{1}{2}\gamma z)^{-1}[-(1 + \gamma)\kappa^{-1}z^2 - (\kappa^2 + 1)\kappa^{-1} - (1 - \tfrac{1}{2}\gamma z)e^{-z}]$$

Furthermore, let

$$K_n^0 = (\pi\gamma)^{1/2} \frac{\Gamma(1 + \pi^{-1}\delta_n)}{\Gamma(\frac{1}{2} + \pi^{-1}\delta_n)}, \quad X_n^0 = \exp\left\{\frac{\delta_n}{\pi} \int_0^{\infty} \ln G_0(x) \frac{dx}{x^2 + \delta_n^2}\right\}$$

$$G_n = \gamma e_n[\tfrac{1}{2}\kappa\sin 2\delta_n + \kappa\delta_n\cos 2\delta_n + 2\delta_n + 2\delta_n e_n(\tfrac{1}{2}\kappa\sin 2\delta_n + \delta_n)^2] \qquad (1.22)$$

$$e_n = (\kappa_+^2 - \delta_n^2 - \kappa\sin^2\delta_n)^{-1}$$

If we compute the integrals in (1.21) by using the theory of residues we find the following formula for the contact stresses

$$\tau(x) = \frac{T}{b} \sum_{n=1}^{\infty} \frac{K_n^0 X_n^0}{G_n} \left(\frac{p_0}{\delta_n} + \sum_{m=1}^{\infty} \frac{A_m}{\beta_m - \delta_n}\right) e^{\delta_n (a - |x|)/b}, \quad |x| > a$$

By virtue of the Tauber-type theorem, (1.15) implies the continuity of the tangential stresses $\tau(x)$ at the points $x = \pm a$, namely, $\tau(\pm a) = \mu p$.

2. THE PROBLEM OF THE PRESSURE OF A FINITE STIFFENER ON AN ELASTIC LAYER WITH FRICTION AND ADHESION

Let us consider the plane deformation of the elastic layer $\{-\infty < x < +\infty,\ -b < y < 0\}$ linked to a rigid foundation and stiffened by the finite stringer $\{-a < x < a,\ -b < y < 0\}$ (Fig. 2). The constants of elasticity of the layer and the stiffener are the same as in Sec. 1. The stiffener is impressed into the layer by a normal uniform load of intensity p and is stretched by forces P_1 and P_2 concentrated at the tips. The contact area is divided into the adhesion zone $\{-c_1 < x < c_2,\ y = 0\}$ and two zones $\{-a_1 < x < c_1,\ y = 0\}$ and $\{c_2 < x < a,\ y = 0\}$ in which Coulomb friction acts. Let us find the points $-c_1$ and c_2 and the contact tangential stresses $\tau(x)$ along the line $-c_1 < x < c_2$.

The expression for the axial deformation of the stringer corresponding to (1.1) has the form

$$\epsilon_x^0(x) = \frac{1 - \nu_0^2}{h E_0} \left[\int_{-a}^{x} \tau(\xi) d\xi + P_1 \right] \tag{2.1}$$

in the case considered. For the stringer to be in equilibrium it is necessary that the condition

$$\int_{-a}^{a} \tau(x) dx = P_2 - P_1 \tag{2.2}$$

should be satisfied.

In the case when

$$\sigma_y = \begin{cases} 0, & |x| > a \\ -p, & |x| < a \end{cases}, \quad \tau_{xy} = \begin{cases} 0, & |x| > a \\ \tau(x), & |x| < a \end{cases}$$

the solution of problem (1.3) for the layer, by virtue of (1.4), has the form

$$\epsilon_x(x, 0) = -\frac{2(1 - \nu^2)}{bE} \int_{-a}^{a} S\left(\frac{x - \xi}{b}\right) \tau(\xi) d\xi - \frac{(1 + \nu)p}{2\pi E} \int_{-\infty}^{\infty} g_0(\alpha) e^{-i\alpha x/b} d\alpha$$
$$g_0(\alpha) = g_1(\alpha) \frac{e^{i\alpha\lambda_0} - e^{-i\alpha\lambda_0}}{i\alpha}, \quad g_1(\alpha) = \frac{\kappa\kappa\ \mathrm{sh}^2\alpha - \alpha^2}{\kappa\,\mathrm{sh}^2\alpha + \alpha^2 + \kappa_+^2}, \quad \lambda_0 = \frac{a}{b} \tag{2.3}$$

where $S(t)$ is the function specified in (1.8). If we substitute expressions (2.1) and (2.3) into relationship (1.5), change the variables and introduce the notation

$$t = x/b + \lambda_1, \quad \eta = \xi/b + \lambda_2, \quad \lambda_1 = c_1/b, \quad \lambda_2 = c_2/b$$
$$\chi(t) = b\tau(-c_1 + bt) \tag{2.4}$$

we obtain the Prandtl-type integro-differential equation

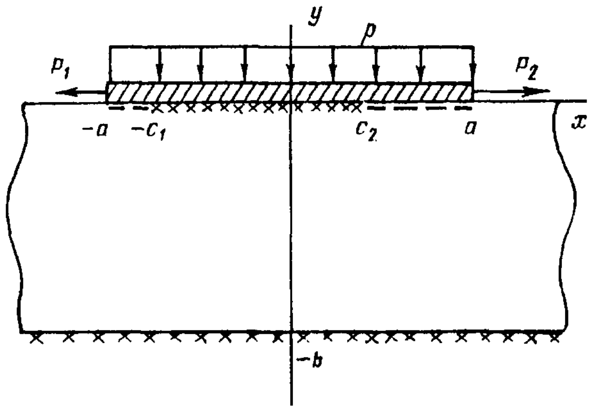

FIG. 2.

$$\int_{\lambda_1-\lambda_0}^{t}\chi(\eta)d\eta+\gamma\int_{\lambda_1-\lambda_0}^{\lambda_1+\lambda_0}S(t-\eta)\chi(\eta)d\eta=-P_1-\frac{\gamma_0}{2\pi}\int_{-\infty}^{\infty}g_0(\alpha)e^{i\alpha\lambda_1}e^{-i\alpha t}d\alpha$$

$$0<t<\lambda,\ \lambda=\lambda_1+\lambda_2,\ \gamma_0=\tfrac{1}{2}pb\gamma(1-\nu)^{-1} \tag{2.5}$$

where γ is the parameter specified in (1.8).

Since the conditions of Coulomb friction are satisfied in the zones $-a<x<-c_1$ and $c_2<x<a$, we have

$$\tau(x)=-\mu p,\ -a<x<-c_1;\quad \tau(x)=\mu p,\ c_2<x<a \tag{2.6}$$

(μ is the coefficient of friction). The different signs in front of μp show that slippage occurs in the opposite directions when x belongs to the zones $(-a,\ c_1)$ or $(c_2,\ a)$. Let us extend the validity of Eq. (2.5) over the whole real axis. To do so we introduce the functions (T) and

$$\chi_0(\eta)=\begin{cases}\chi(\eta),\ 0<\eta<\lambda\\ -\sigma,\ \lambda_1-\lambda_0<\eta<0\\ \sigma,\ \lambda<\eta<\lambda_0+\lambda_1\\ 0,\ \eta\notin(\lambda_1-\lambda_0,\ \lambda_0+\lambda_1)\end{cases}\qquad \sigma=b\mu p,\quad f_0(t)=\begin{cases}1,\ 0<t<\lambda\\ 0,\ t\notin(0,\lambda)\end{cases} \tag{2.7}$$

We then obtain

$$\int_{\lambda_1-\lambda_0}^{t}\chi_0(\eta)d\eta+\gamma\int_{-\infty}^{\infty}S(t-\eta)\chi_0(\eta)d\eta=-f_0(t)P_1-\frac{\gamma_0}{2\pi}\int_{-\infty}^{\infty}g_0(\alpha)e^{i\alpha\lambda_1}e^{-i\alpha t}d\alpha+$$
$$+\chi_-(t)+\chi_+(t),\ -\infty<t<\infty$$

Let us use notation (1.10) for the functions $\Phi_1^{\pm}(\alpha)$ and take into account the Fourier integrals

$$\Phi_2^+(\alpha)=\int_0^{\infty}\chi_+(\lambda+\xi)e^{i\alpha\xi}d\xi,\ \Phi_2^-(\alpha)=\int_{-\infty}^{\infty}\chi_-(\xi)e^{i\alpha\xi}d\xi$$

By virtue of (2.7) and (2.2), we have

$$V\left[\int_{\lambda_1-\lambda_0}^{t}\chi_0(\eta)d\eta\right]=\frac{1}{i\alpha}\{e^{i\alpha(\lambda_1+\lambda_0)}(P_2-P_1)-\Phi_1^+(\alpha)+$$
$$+(i\alpha)^{-1}\sigma[1-e^{i\alpha(\lambda_1-\lambda_0)}-e^{i\alpha(\lambda_1+\lambda_0)}+e^{i\alpha\lambda}]\}$$

In a manner similar to that of Sec. 1 we obtain the vectorial Riemann problem

$$G(\alpha)\,[\Phi_1^+(\alpha) - (i\alpha)^{-1}\sigma(1 - e^{i\alpha(\lambda_1-\lambda_0)} - e^{i\alpha(\lambda_1+\lambda_0)} + e^{i\alpha\lambda})] =$$
$$= -i\alpha[\Phi_2^-(\alpha) + e^{i\alpha\lambda}\Phi_2^+(\alpha)] + i\alpha e^{i\alpha(\lambda_1+\lambda_0)}(P_2 - P_1) + (e^{i\alpha\lambda}-1)P_1 +$$
$$+ \gamma_0 g_0(\alpha)e^{i\alpha\lambda_1}i\alpha,\quad \Phi_1^+(\alpha) = e^{i\alpha\lambda}\Phi_1^-(\alpha),\quad -\infty<\alpha<+\infty \tag{2.8}$$

We take into account formulae (1.12) and introduce the functions

$$\varphi_1^+(\alpha) = \Phi_1^+(\alpha) + (i\alpha)^{-1}\sigma(e^{i\alpha(\lambda_1+\lambda_0)} - e^{i\alpha\lambda})$$
$$\varphi_1^-(\alpha) = \Phi_1^-(\alpha) + (i\alpha)^{-1}\sigma(e^{-i\alpha(\lambda_0+\lambda_2)} - e^{-i\alpha\lambda})$$
$$\varphi_2^+(\alpha) = -i\alpha\Phi_2^+(\alpha) + e^{i\alpha(\lambda_0-\lambda_2)}(P_2 - P_1) + P_1,\quad \varphi_2^-(\alpha) = -i\alpha\Phi_2^-(\alpha) - P_1$$
$$\Psi^\pm(\alpha) = \sum_{n=1}^{\infty}\frac{iA_n^\pm}{\alpha\pm i\beta_n},\quad \Omega^\pm(\alpha) = \sum_{n=1}^{\infty}\frac{iB_n^\pm}{\alpha\pm i\delta_n} \tag{2.9}$$

The quantities $\pm i\beta_n$ are the poles of the functions $G(\alpha)$ and $g_1(\alpha)$, but $\pm i\delta_n$ are zeros of the function $G(\alpha)$ that were considered in Sec. 1. The coefficients $A_n^\pm$ and $B_n^\pm$ will be evaluated later. We rewrite problem (2.8) in the form

$$K^+(\alpha)X^+(\alpha)\,\{\varphi_1^+(\alpha) - (i\alpha)^{-1}\sigma - e^{i\alpha\lambda}[G(\alpha)]^{-1}[\varphi_2^+(\alpha) + \gamma_0 g_1(\alpha)e^{i\alpha(\lambda_0-\lambda_2)}]\} +$$
$$+ (i\alpha)^{-1}\sigma - \Omega^-(\alpha) - \Psi^+(\alpha) = [K^-(\alpha)X^-(\alpha)]^{-1}\,\{\varphi_2^-(\alpha) - e^{-i\alpha(\lambda_0-\lambda_1)}[(i\alpha)^{-1}\sigma G(\alpha) +$$
$$+ \gamma_0 g_1(\alpha)]\} + (i\alpha)^{-1}\sigma - \Omega^-(\alpha) - \Psi^+(\alpha)$$
$$K^-(\alpha)X^-(\alpha)\,\{\varphi_1^-(\alpha) - (i\alpha)^{-1}\sigma - e^{-i\alpha\lambda}[G(\alpha)]^{-1}[\varphi_2^-(\alpha) - \gamma_0 g_1(\alpha)e^{-i\alpha(\lambda_0-\lambda_1)}]\} +$$
$$+ (i\alpha)^{-1}\sigma - \Omega^+(\alpha) - \Psi^-(\alpha) = [K^+(\alpha)X^+(\alpha)]^{-1}\,\{\varphi_2(\alpha) + e^{i\alpha(\lambda_0-\lambda_2)}[-(i\alpha)^{-1}\sigma G(\alpha) +$$
$$+ \gamma_0 g_1(\alpha)]\} + (i\alpha)^{-1}\sigma - \Omega^+(\alpha) - \Psi^-(\alpha) \tag{2.10}$$

Let us choose the coefficients $A_n^\pm$ in such a way that the functions $\Psi^+(\alpha)$ and $\Psi^-(\alpha)$ eliminate the poles of the right-hand sides of equalities (2.10). We then have $(n=1, 2, \dots)$

$$\operatorname*{res}_{\alpha=-i\beta_n}\{[K^-(\alpha)X^-(\alpha)]^{-1}e^{-i\alpha(\lambda_0-\lambda_1)}[(i\alpha)^{-1}\sigma G(\alpha) + \gamma_0 g_1(\alpha)] - \Psi^+(\alpha)\} = 0$$
$$\operatorname*{res}_{\alpha=i\beta_n}\{[K^+(\alpha)X^+(\alpha)]^{-1}e^{i\alpha(\lambda_0-\lambda_2)}[-(i\alpha)^{-1}\sigma G(\alpha) + \gamma_0 g_1(\alpha)] - \Psi^-(\alpha)\} = 0$$

whence we find the explicit formulae for $A_n^\pm$

$$A_n^+ = e^{-\beta_n(\lambda_0-\lambda_1)}q_n,\quad A_n^- = e^{-\beta_n(\lambda_0-\lambda_2)}q_n$$
$$q_n = \frac{1}{2K_nX_n}\left(\gamma\sigma + \gamma_0\frac{\kappa\kappa_-\sin^2\beta_n - \beta_n^2}{\frac{1}{2}\kappa\sin 2\beta_n + \beta_n}\right)$$

The values K_n and X_n were specified in (1.20).

Let the choice of the coefficients $B_n^\pm$ ensure the validity of the principle of continuity. According to Liouville's theorem and because of the fact that the stresses $\tau(x)$ are bounded at the points $x=-c_1$ and $x=c_2$ we then find the solution of problem (2.8)

$$\Phi_1^+(\alpha) = \frac{\omega_1(\alpha)}{K^+(\alpha)X^+(\alpha)} + \frac{e^{i\alpha\lambda}\omega_2(\alpha)}{K^-(\alpha)X^-(\alpha)} + \frac{\sigma(1+e^{i\alpha\lambda})}{i\alpha},\quad \Phi_1^-(\alpha) = e^{-i\alpha\lambda}\Phi_1^+(\alpha) \tag{2.11}$$

$$\Phi_2^+(\alpha) = -(i\alpha)^{-1}\,\{K^+(\alpha)X^+(\alpha)\omega_2(\alpha) + e^{i\alpha(\lambda_0-\lambda_2)}[(i\alpha)^{-1}\sigma G(\alpha) -$$
$$- \gamma_0 g_1(\alpha) + P_1 - P_2] - P_1\}$$
$$\Phi_2^-(\alpha) = -(i\alpha)^{-1}\,\{K^-(\alpha)X^-(\alpha)\omega_1(\alpha) + e^{-i\alpha(\lambda_0-\lambda_1)}[(i\alpha)^{-1}\sigma G(\alpha) + \gamma_0 g_1(\alpha)] + P_1\}$$
$$\omega_1(\alpha) = -(i\alpha)^{-1}\sigma + \Omega^-(\alpha) + \Psi^+(\alpha),\quad \omega_2(\alpha) = -(i\alpha)^{-1}\sigma + \Omega^+(\alpha) + \Psi^-(\alpha)$$

from (2.10) and (2.9).

The functions $\Phi_2^+(\alpha)$ and $\Phi_2^-(\alpha)$ are analytic at the point $\alpha = 0$ if and only if the conditions

$$\begin{aligned} &\{K^+(\alpha)X^+(\alpha)\omega_2(\alpha) + e^{i\alpha(\lambda_0-\lambda_2)}[(i\alpha)^{-1}\sigma G(\alpha) - \gamma_0 g_1(\alpha) + P_1 - P_2] - P_1\}_{\alpha=0} = 0 \\ &\{K^-(\alpha)X^-(\alpha)\omega_1(\alpha) + e^{-i\alpha(\lambda_0-\lambda_1)}[(i\alpha)^{-1}\sigma G(\alpha) + \gamma_0 g_1(\alpha)] + P_1\}_{\alpha=0} = 0 \end{aligned} \tag{2.12}$$

hold. Taking formulae (1.17) into account we can rewrite equalities (2.12) in the form

$$\begin{aligned} &\lambda_1 = d_* - \sigma^{-1}(P_1 + A_*^+ - B_*^-) \\ &\lambda_2 = d_* - \sigma^{-1}(P_2 + A_*^- - B_*^+) \\ &d_* = d_0 + \frac{2}{\pi}\ln 2 + \lambda_0, \; A_*^\pm = \sum_{n=1}^{\infty}\frac{A_n^\pm}{\beta_n}, \quad B_*^\pm = \sum_{n=1}^{\infty}\frac{B_*^\pm}{\delta_n} \end{aligned} \tag{2.13}$$

Equalities (2.13) ensure not only that the functions $\Phi_2^\pm(\alpha)$ are analytic but constitute the non-linear system for finding the parameters λ_1 and λ_2.

System (2.13) also implies the condition of equilibrium (2.2).

Actually, if we take into account relations (2.4), (2.7) and (1.10) we can write condition (2.2) in the form

$$\{\Phi_1^+(\alpha) - (i\alpha)^{-1}\sigma[1 - e^{i\alpha(\lambda_1-\lambda_0)} - e^{i\alpha(\lambda_1+\lambda_0)} + e^{i\alpha\lambda}]\}_{\alpha=0} = P_2 - P_1$$

which, by virtue of (2.11) and (1.17), is equivalent to the following condition

$$A_*^+ - A_*^- + B_*^+ - B_*^- + \sigma(\lambda_1 - \lambda_2) = P_2 - P_1$$

but this equality is obtained from (2.13) by subtracting the second equation of (2.13) from the first.

We will now choose the coefficients $B_n^\pm$. For the functions $\Phi_1^\pm(\alpha)$ to be analytic at the points $\alpha = \pm i\delta_n \in C^\pm$ the infinite system of Poincaré–Koch algebraic equations

$$\begin{aligned} &B_n^\pm = e^{-\delta_n\lambda}\Delta_n^0\left(f_n^\pm + \sum_{m=1}^{\infty}\frac{B_m^\mp}{\delta_n+\delta_m}\right) \quad (n = 1, 2, \ldots) \\ &\Delta_n^0 = \frac{(K_n^0 X_n^0)^2}{G_n}, \quad f_n^\pm = \frac{\sigma}{\delta_n} + \sum_{m=1}^{\infty}\frac{A_m^\pm}{\delta_n - \beta_m} \end{aligned} \tag{2.14}$$

must be satisfied.

The quantities K_n^0, X_n^0 and G_n were specified in (1.22). System (2.14) may be effectively solved using the recurrent relations

$$\begin{aligned} &B_n^\pm = e^{-\delta_n\lambda}\sum_{k=0}^{\infty} b_{nk}^\pm, \quad b_{n0}^\pm = \Delta_n^0 f_n^\pm \\ &b_{np}^\pm = \Delta_n^0 \sum_{j=1}^{p}\frac{e^{-\lambda\delta_j}}{\delta_n + \delta_j} b_{j,p-j}^\mp \end{aligned}$$

Let us obtain formulae for the contact stresses

$$\tau(x) = \frac{1}{b}\chi\left(\frac{x}{b} + \lambda_1\right), \quad \chi(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\Phi_1^+(\alpha)e^{-i\alpha t}d\alpha \tag{2.15}$$

If we substitute (2.11) into (2.15) and use Cauchy's theorem we find

$$\tau(x) = \frac{1}{b} \sum_{n=1}^{\infty} \frac{1}{K_n^0 X_n^0} \left[B_n^- e^{\delta_n (x+c_1)/b} - B_n^+ e^{\delta_n(-x+c_2)/b}\right], \quad -c_1 < x < c_2 \tag{2.16}$$

In the case when $P_1 = P_2$ the relations

$$c_1 = c_2 = c, \quad \lambda_1 = \lambda_2 = \tfrac{1}{2}\lambda$$

$$f_n^{\pm} = f_n = \frac{\sigma}{\delta_n} + \sum_{m=1}^{\infty} \frac{A_m}{\delta_n - \delta_m}, \quad A_m^{\pm} = A_m$$

hold and, as a result, the infinite system (2.14) with respect to $B_m^+ = B_m^- = B_m$ is simplified

$$B_n = e^{-\delta_n \lambda} \Delta_n^0 \left(f_n + \sum_{m=1}^{\infty} \frac{B_m}{\delta_n + \delta_m}\right)$$

and formula (2.16) for the contact stresses becomes

$$\tau(x) = \frac{2}{b} \sum_{n=1}^{\infty} \frac{B_n}{K_n^0 X_n^0} e^{\frac{1}{2}\lambda\delta_n} \operatorname{sh} \frac{\delta_n x}{b}, \quad -c < x < c \tag{2.17}$$

By virtue of (2.6), formulae (2.16) and (2.17) lead to the conclusion that the tangential stresses $\tau(x)$ are continuous at the points $x = -c_1$ and $x = c_2$ and are bounded at the tips $x = \pm a$ of the stiffener.

3. NUMERICAL COMPUTATIONS

For all actual values of the parameters of the problem of the infinite stiffener numerical analysis has revealed the existence and uniqueness of the solution of the non-linear equation (1.18) defining the variable λ. Moreover, the inequality $|\tau_{xy}(x, 0)| < \mu |\sigma_y(x, 0)|$ holds in the adhesion zone. Plots of the contact stresses $T^{-1}\tau(x)$ for $\nu_0 = \nu = 0.3$, $E_0/E = 2$, $b = 1$, $h = 0.01$, $p = 1$ are shown in Fig. 3 for the different values of the coefficient of friction $\mu = 0.1$ (curve 1), $\mu = 0$ (curve 2) and $\mu = 0.5$ (curve 3). Below we present the parameter $\lambda = 2a/b$ as a function of μ for the same values of the parameters of the problem

μ	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
λ	10.62	5.62	3.95	3.12	2.62	2.28	2.04	1.87	1.73

as a function of E_0/E with $\mu = 0.3$ (the other parameters are the same)

E_0/E	0.1	1	2	5	10	100
λ	4.20	4.07	3.95	3.66	3.27	1.32

and as a function of b with $\mu = 0.3$ and $E_0/E = 2$

b	1	2	3	4	5	7	10	15	20
λ	3.95	2.40	1.89	1.63	1.48	1.31	1.18	1.08	1.03

Calculations were also carried out for the problem of the extension of a finite stiffener. Plots of the contact stresses $\tau(x)$ for $\nu_0 = \nu = 0.3$, $E_0/E = 2$, $a = 1$, $b = 1$, $h = 0.01$, $p = 1$, $\mu = 0.3$ are shown in Fig. 4. Curve 1 is for $P_1 = P_2 = 0$ (in this case, $\lambda_1 = \lambda_2 = 0.7656$ and do not depend on p), curve 2 is for

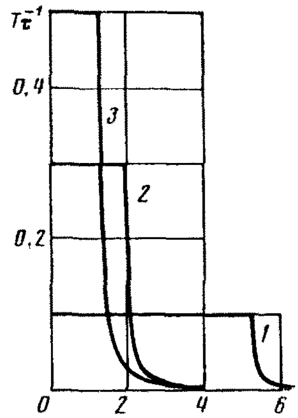

FIG. 3.

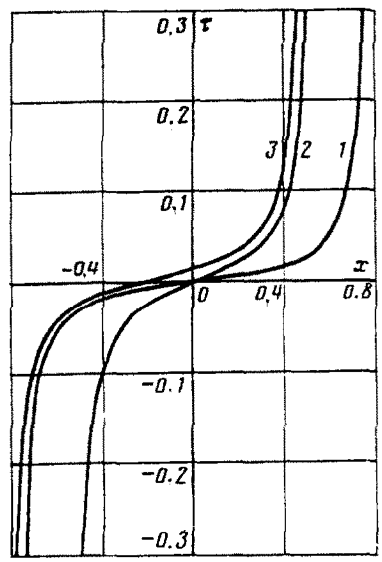

FIG. 4.

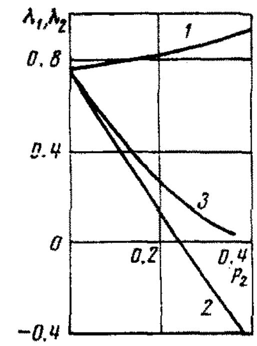

FIG. 5.

$P_1 = P_2 = 0.1$, ($\lambda_1 = \lambda_2 = 0.4945$), and curve 3 is for $P_1 = 0$, $P_2 = 0.1$ ($\lambda_1 = 0.7926$, $\lambda_2 = 0.4592$). λ_1 and λ_2 are plotted against P_2 with $P_1 = 0$ (curves 1 and 2, respectively) and a plot of $\lambda_1 = \lambda_2$ against $P_1 = P_2$ (curve 3) is shown in Fig. 5. The computations were carried out for the same values of the parameters of the problem as in Fig. 4.

In the case $P_1 = P_2$, the non-linear system (2.13) always has the solution $\lambda_1 = \lambda_2$, and moreover, it is unique. If $P_1 \neq P_2$, the quantity P_2 must satisfy the condition $P_2{}^* < P_2 < P_2{}^{**}$ for arbitrary P_1. In particular for the same values of the parameters ν, ν_0, E_0/E, a, b, h, μ and p that were adopted when plotting Figs 4 and 5, system (2.13) is solvable when $P_1 = 0$ if and only if $0 \leqslant P_2 < 0.475$. When $P_2 \approx 0.475$ we have $\lambda_1 = 1$ and $\lambda_2 = -0.65$, but when $P_2 > 0.475$ a value of $\lambda_1 \in (0, 1)$ that satisfies system (2.13) does not exist.

REFERENCES

1. MELAN E., Ein Beitzag zur Theorie geschweisster Verbindungen. *Ing. Arch.* **3**, 1932.
2. ARUTYUNYAN N. Kh., The contact problem for a half-plane with elastic reinforcement. *Prikl. Mat. Mekh.* **32**, 632–646, 1968.
3. ANTIPOV Yu. A. and ARUTYUNYAN N. Kh., Contact problems of the theory of elasticity with friction and adhesion. *Prikl. Mat. Mekh.* **55**, 1005–1017, 1991.
4. VLADIMIROV V. S., *Generalized Functions in Mathematical Physics.* Nauka, Moscow, 1979.
5. UFLYAND Ya. S., *Integral Transformations in Problems of the Theory of Elasticity.* Nauka, Leningrad, 1967.

Translated by A.S.S.